Moment of Inertia —

For rigid body — about the axis of rotation.

[a) mass distributed around the axis]

[b) distance of particles of the body from the axis of rotation].

$$I = mr^2$$

Moment of inertia for n-particle system.

$$I = \sum_{i=1}^{i=n} m_i r_i^2$$

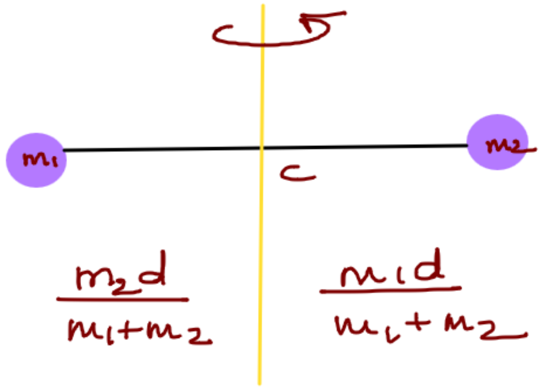
infinitesimally small segment of mass (dm).

moment of inertia for continuous mass distribution.

$$I = \int (dm) r^2$$

1. Two particle system —

seperated by a distance d



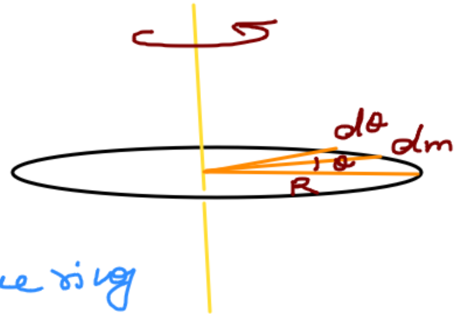
Axis of rotation passes through the centre of mass of the system and is $\perp$ to the line joining the 2 particles.

moment of inertia about the axis:

$$I = m_1\left[\frac{m_2 d}{m_1+m_2}\right]^2 + m_2\left[\frac{m_1 d}{m_1+m_2}\right]^2$$

$$= \frac{m_1 m_2}{m_1+m_2} d^2$$

2. Ring



Ring of mass m & radius R

Axis of rotation passes through the centre of the ring $\perp$ to the plane of ring.

We select a infinitesimally small segment of mass dm anywhere on the ring

$\perp$ distance of the segment from the axis of rotation is R.

moment of inertia.

$$I = \int_{ring} (dm) R^2$$

[R is a constant]

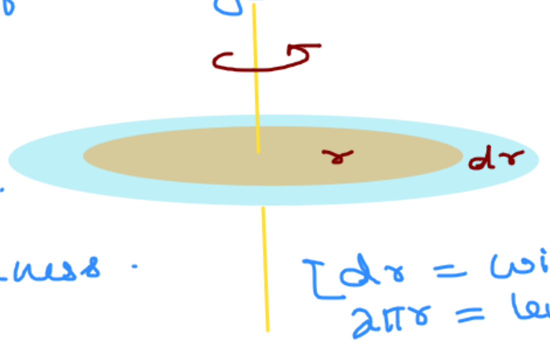
$$= \left[\int_{ring} dm\right] R^2$$

$$= mR^2$$

[Since dm is the total mass of the ring].

3 disc

axis of rotation is passing through the centre of disc is perpendicular to its plane.



dr — infinitesimally small radial thickness.

$$dm = \left(\frac{m}{\pi R^2}\right) 2\pi r\, dr$$

[dr = width
$2\pi r$ = length]

$$dm = \frac{2m}{R^2} r\,dr$$

moment of inertia of ring — $dI = (dm) r^2$

$$\therefore \quad dI = \left(\frac{2m}{R^2} r\,dr\right) r^2$$

$$= \frac{2m}{R^2} r^3 dr$$

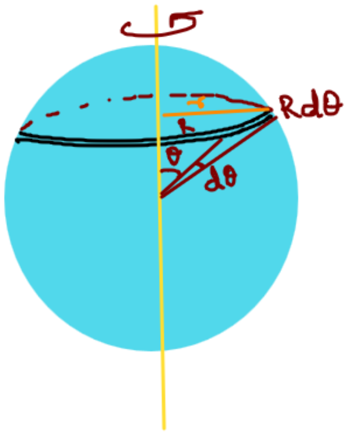
$$I = \int dI = \frac{2m}{R^2} \int_{r=0}^{r=R} r^3 dr$$

$$I = \frac{2m}{R^2}\left[\frac{R^4}{4}\right]_0^R$$

$$= \frac{2m}{R^2}\left[\frac{R^4}{4} - 0\right]$$

$$I = \frac{1}{2} m R^2$$

4. Hollow sphere / spherical shell —



axis of rotation passing through the centre of the spherical shell.

θ = angle

$d\theta$ = angular thickness

radius of the ring segment

$$r = R\sin\theta$$

area of the ring segment

$$dA = (2\pi r)(R\,d\theta)$$

$$= (2\pi R\sin\theta)(R\,d\theta)$$

$$= 2\pi R^2 \sin\theta\, d\theta$$

[$2\pi r$ = length
$R\,d\theta$ = width]

mass of the ring segment —

$$dm = \frac{m}{4\pi R^2}(dA)$$

$$= \frac{m}{4\pi R^2}(2\pi R^2 \sin\theta\, d\theta)$$

$$dm = \frac{M}{2}\sin\theta\, d\theta$$

$\therefore$ moment of inertia of ring segment

$$dI = (dm)r^2$$

$$= \left(\frac{M}{2}\sin\theta\, d\theta\right)(R\sin\theta)^2$$

$$= \frac{mR^2}{2}\sin^3\theta\, d\theta$$

$$I = \frac{mR^2}{2}\int_{\theta=0}^{\theta=\pi}\sin^3\theta\, d\theta$$

$$= \frac{mR^2}{2}\int_{\theta=0}^{\theta=\pi}(1-\cos^2\theta)(\sin\theta\, d\theta) \quad \text{—— (1)}$$

Let $t = \cos\theta$

$$dt = -\sin\theta\, d\theta$$

$$\theta = 0 \Rightarrow t = \cos 0 = 1$$

$$\theta = \pi \Rightarrow t = \cos\pi$$
$$= -1$$

Using eq (1)

$$I = \frac{mR^2}{2}\int_{1}^{-1}(1-t^2)(-dt)$$

adjusting -ve sign:

$$I = \frac{mR^2}{2}\int_{-1}^{1}(1-t^2)\,dt$$

$$= \left(\frac{mR^2}{2}\right)\left(2\int_{0}^{1}(1-t^2)\,dt\right)$$

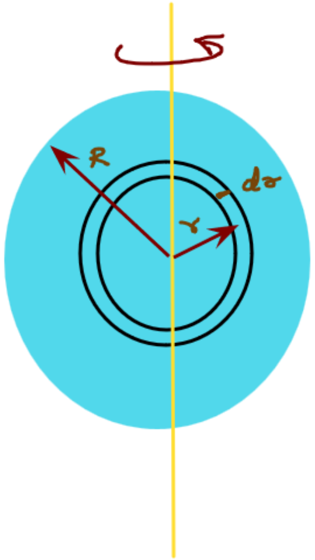
$$= mR^2\int_{0}^{1}(1-t^2)\,dt$$

$$= mR^2\left[t-\frac{t^3}{3}\right]_0^1$$

$$= mR^2\left[1-\frac{1}{3}\right]$$

$$I = \frac{2}{3} m R^2$$

5. <u>Solid sphere</u>



thickness $= dr$

mass of the sphere is uniformly distributed over the volume.

Volume of segment.

$$dV = 4\pi r^2 dr$$

mass of segment

$$dm = \frac{m}{\frac{4}{3}\pi R^3}(4\pi r^2 dr)$$

$$= \frac{3m}{R^3} r^2 dr$$

moment of inertia of the segment which is in the form of a spherical shell

$$dI = \frac{2}{3}(dm) r^2$$

$$dI = \frac{2}{3}\left(\frac{3m}{R^3} r^2 dr\right) r^2$$

$$= \frac{2m}{R^3} r^4 dr$$

Total moment of inertia of the solid sphere $I = \int dI = \frac{2m}{R^3} \int_{r=0}^{r=R} r^4 dr$

$$I = \frac{2m}{R^3} \left[\frac{R^5}{5}\right]_0^R$$

$$= \frac{2m}{R^3} \left[\frac{R^5}{5} - 0\right]$$

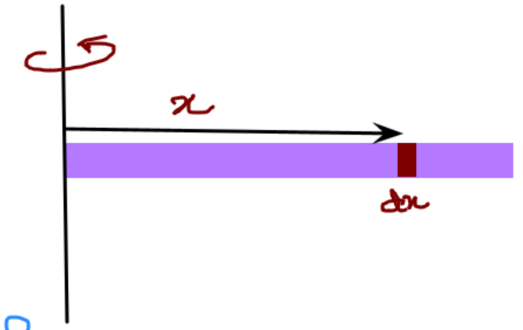
$$= \frac{2}{5} MR^2$$

6. <u>Thin rod</u>

mass $= m$

length $= L$ with negligible cross section.

Axis of rotation is passing through the end of the rod & $\perp$ to the length of the rod.



We select a segment of length dx at a distance x from end of the rod [as in fig].

mass is uniformly distributed over the length of the rod.

mass of the segment

$$dm = \frac{m}{L} dx$$

segment is like a particle.

its moment of inertia

$$dI = (dm)x^2$$

$$= \left(\frac{m}{L} dx\right) x^2$$

Calculating the total moment of inertia of the rod.

$$I = \frac{m}{L}\int_0^L x^2 dx$$

$$= \frac{m}{L}\left(\frac{x^3}{3}\right)_0^L$$

$$I = \frac{m}{L}\left(\frac{L^3}{3} - 0\right)$$

$$= \frac{1}{3} mL^2$$

If axis of rotation passes through the centre of rod and is $\perp$ to its length then we consider the given rod as 2 rods of mass $\frac{m}{2}$ each and length $\frac{L}{2}$.

$\therefore$ moment of inertia

$$I = 2\left[\frac{1}{3}\left(\frac{m}{2}\right)\left(\frac{L}{2}\right)^2\right]$$

$$I = \frac{1}{12} mL$$

<u>Perpendicular axis theorem</u> —

This theorem is valid only for laminar objects.

[Laminar objects are flat objects with negligible thickness like thin disc, ring & objects like door panels.

Thickness of the object is very small

$\therefore$ we assume all its particles lying on the X-Y plane only.

mass = m_i (x_i, y_i)

$\perp$ distance of this particle from X-axis is y_i

$\therefore$ moment of inertia of object about X-axis

$$I_x = \sum_{i=1}^{i=n} m_i y_i^2$$

$\perp$ distance of the above particle from Y-axis is x_i

$\therefore$ moment of inertia of object about Y-axis.

$$I_y = \sum_{i=1}^{i=n} m_i x_i^2$$

$\perp$ distance of the same particle from Z axis is $\sqrt{x_i^2+y_i^2}$

$\therefore$ moment of inertia of object about Z-axis

$$I_z = \sum_{i=1}^{i=n} m_i \left[\sqrt{x_i^2+y_i^2}\,\right]^2$$

$$I_z = \sum_{i=1}^{i=n} m_i y_i^2 + \sum_{i=1}^{i=n} m_i x_i^2$$

$$\boxed{I_z = I_x + I_y}$$ Perpendicular axis theorem.

This theorem states that —

Sum of the moments of inertia of a laminar object about two mutually $\perp$ axes lying on its plane is equal to the moment of inertia of the object about an axis $\perp$ to its plane and that is passing through the point of intersection of other two axes lying on the plane of the object.